Computation of solid angles and form factors

Exercise UB10, November 2012, UTC, Professor: Benoit Beckers

Compare two enclosures of same volume (500 m^3) and height (5 m). Their bases are correspondingly a square and a circle. Perform the comparison by filling the following table.

Enclosure	Square base	Circular base
Envelope area	?	?
Solid angle of the roof as seen from the center of the floor	? Indication: consider the solid angle of a face of a regular polyhedron as seen from its center	? Indication: compute the area of the spherical cap of the sphere centered in the center of the base and limited by its contour on the roof
Differential form factor of the roof as seen from the center of the floor	? Indication: use Lambert[1] formula	? Indication: use Nusselt[2] analogy

Useful theoretical background

1. Without obstruction, the differential form factor of a polygonal surface *P*, *Q*, *R*… is obtained by the **Lambert's exact formula**:

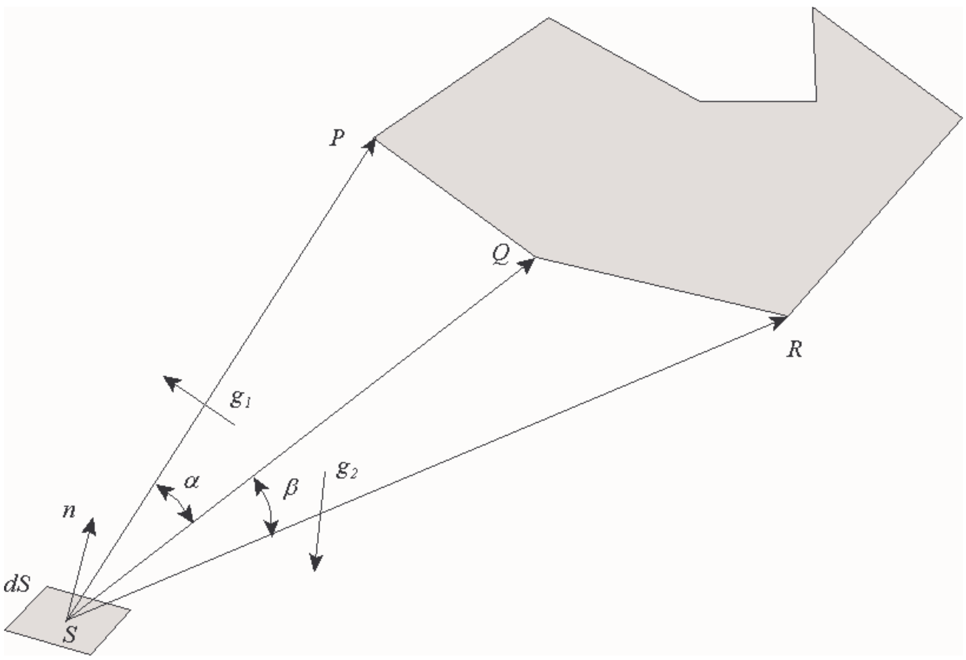


Figure 1 : *Point (S) to area (PQR…) view factor*[3]

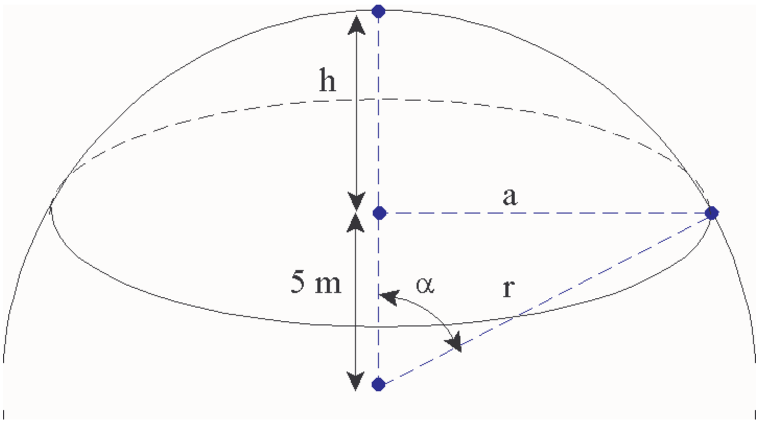
[1] Johann Heinrich Lambert, "Photometria sive de mensura et gradibus luminis, colorum et umbrae", 1760, German translation by E. Anding in Ostwald's Klassiker der Exakten Wissenschaften, Vol. 31-33, Leipzig, 1892. Cited by Peter Schröder & Pat Hanrahan, « A Closed Form Expression for the Form Factor between Two Polygons », Research Report CS-TR-404-93, January 1993.

[2] W. Nusselt, "Graphische bestimmung des winkelverhaltnisses bei der wärmestrahlung", Zeitschrift des Vereines Deutscher Ingenieure, 72(20):673 1928. See: B. Beckers, L. Masset & P. Beckers, "Commentaires sur l'analogie de Nusselt", Rapport Helio_004_fr, 2009, http://www.heliodon.net/.

$$F_{dS-j} = \frac{1}{2\pi}\sum_j n.g_j \qquad (1.1)$$

Vector n is normal to the surface supporting dS and for which we calculate the form factor. Vectors g_j are normal to the faces SPQ, etc. of the pyramid. Their modules are equal to the apex angles of the faces.

2. Area of a spherical cap



According to the notations of the above figure, the area of a spherical cap is easily obtained, either in function of the angle α or in function of the radius r of the sphere and the distance h from the base of the cap to its pole.

$$Area_{cap} = 2\pi r^2\left(1-\cos\alpha\right) = 2\pi rh \qquad (1.2)$$

Solution

Enclosure	Square base	Circular base
Envelope area	Walls and roof: **300** m^2 Floor: **100** m^2 Total : **400** m^2	$a = \sqrt{100/\pi}$ = **5.642** m Wall: **177.2454** m^2 Total: **377.2454** m^2
Solid angle of the roof as seen from the center of the floor	Solid angle = 1/3 = **0.3333** Expressed as a fraction of the hemisphere solid angle	$r = \sqrt{25+100/\pi}$ = 7.5386 m $h = r - 5$ Solid angle = h / r = **0.3367** (fraction of the hemisphere solid angle)
Differential form factor of the roof as seen from the center of the floor	**0.5541** Expressed as a fraction of the hemisphere form factor	100 / (πr^2) = a^2 / r^2 = **0.5601** Expressed as a fraction of the hemisphere form factor

[3] P. Beckers & B. Beckers, *Radiative Simulation Methods*, in **Solar Energy at Urban Scale**, chapter 10, Ed. B. Beckers, John Wiley and Sons, Inc., **2012**.

Some details of the solution

The differential form factor of the square roof *PQRT* as seen from the center *S* of the floor is easily by directly applying the Lambert's formula (1.1):

$$F = \frac{1}{2\pi}\left[\begin{array}{l} ar\cos\left(\frac{\overrightarrow{SP}\cdot\overrightarrow{SQ}}{\left|\overrightarrow{SP}\right|\left|\overrightarrow{SQ}\right|}\right)\frac{\overrightarrow{SP}\times\overrightarrow{SQ}}{\left|\overrightarrow{SP}\times\overrightarrow{SQ}\right|}\cdot\vec{n} + ar\cos\left(\frac{\overrightarrow{SQ}\cdot\overrightarrow{SR}}{\left|\overrightarrow{SQ}\right|\left|\overrightarrow{SR}\right|}\right)\frac{\overrightarrow{SQ}\times\overrightarrow{SR}}{\left|\overrightarrow{SQ}\times\overrightarrow{SR}\right|}\cdot\vec{n} + \\ ar\cos\left(\frac{\overrightarrow{SR}\cdot\overrightarrow{ST}}{\left|\overrightarrow{SR}\right|\left|\overrightarrow{ST}\right|}\right)\frac{\overrightarrow{SR}\times\overrightarrow{ST}}{\left|\overrightarrow{SR}\times\overrightarrow{ST}\right|}\cdot\vec{n} + ar\cos\left(\frac{\overrightarrow{ST}\cdot\overrightarrow{SP}}{\left|\overrightarrow{ST}\right|\left|\overrightarrow{SP}\right|}\right)\frac{\overrightarrow{ST}\times\overrightarrow{SP}}{\left\|\overrightarrow{ST}\times\overrightarrow{SP}\right\|}\cdot\vec{n} \end{array}\right] \tag{1.3}$$

By symmetry it is sufficient to compute a single term (one face of the pyramid):

$$F = \frac{2}{\pi}\left[ar\cos\left(\frac{\overrightarrow{SP}\cdot\overrightarrow{SQ}}{\left|\overrightarrow{SP}\right|\left|\overrightarrow{SQ}\right|}\right)\frac{\overrightarrow{SP}\times\overrightarrow{SQ}}{\left|\overrightarrow{SP}\times\overrightarrow{SQ}\right|}\cdot\vec{n}\right] \tag{1.4}$$

Finally, simple calculations lead to the result:

$$\begin{array}{l} \overrightarrow{SP} = \begin{bmatrix}5 & -5 & 5\end{bmatrix} \;;\; \left|\overrightarrow{SP}\right| = \sqrt{75} \\ \overrightarrow{SQ} = \begin{bmatrix}5 & 5 & 5\end{bmatrix} \;;\; \left|\overrightarrow{SQ}\right| = \sqrt{75} \\ \overrightarrow{SP}\cdot\overrightarrow{SQ} = 25 \;;\; \alpha = ar\cos\left(\frac{1}{3}\right) = 1.2310\ radians = 70.5288° \\ \left(\overrightarrow{SP}\times\overrightarrow{SQ}\right)\cdot\vec{n} = 50 \;;\; \left|\overrightarrow{SP}\times\overrightarrow{SQ}\right| = \sqrt{5000} \\ F = \frac{2}{\pi}1.2310\frac{50}{\sqrt{5000}} = 0.5541 \end{array} \tag{1.5}$$